Example 2.1 The following is a 5×5 Latin square for data taken from a manurial experiment with sugarcane. The five treatments were as follows :

A : no manure,

B : an inorganic manure,

C, D and E : three levels of farm-yard manure.

TABLE 2.4

PLAN AND YIELD OF SUGARCANE (IN SUITABLE UNITS) PER PLOT

Row	Column				
	I	II	III	IV	V
I	*A* 52·5	*E* 46·3	*D* 44·1	*C* 48·1	*B* 40·9
II	*D* 44·2	*B* 42·9	*A* 51·3	*E* 49·3	*C* 32·6
III	*B* 49·1	*A* 47·3	*C* 38·1	*D* 41·0	*E* 47·2
IV	*C* 43·2	*D* 42·5	*E* 67·2	*B* 55·1	*A* 45·3
V	*E* 47·0	*C* 43·2	*B* 46·7	*A* 46·0	*D* 43·2

Analyse the above data to find out if there are any treatment effects.

The five row totals are : 231·9, 220·3, 222·7, 253·3 and 226·1 ; the five column totals are : 236·0, 222·2, 247·4, 239·5 and 209·2 ; the five treatment totals are : 242·4, 234·7, 205·2, 215·0 and 257·0.

The grand total is 1,154·3.

The correction factor $= \dfrac{(1{,}154\cdot3)^2}{25} = 53{,}296\cdot3333.$

$$\text{Total } SS = (52\cdot5)^2+(46\cdot3)^2+\ldots+(46\cdot0)^2+(43\cdot2)^2-53{,}296\cdot3333$$
$$= 54{,}273\cdot51 - 53{,}296\cdot3333 = 977\cdot1767.$$

$$\text{Row } SS = \frac{(231\cdot9)^2+(220\cdot3)^2+(222\cdot7)^2+(253\cdot3)^2+(226\cdot1)^2}{5} - 53{,}296\cdot3333$$
$$= \frac{267{,}187\cdot09}{5} - 53{,}296\cdot3333 = 53{,}437\cdot4180 - 53{,}296\cdot3333$$
$$= 141\cdot0847.$$

$$\text{Column } SS = \frac{(236\cdot0)^2+(222\cdot2)^2+(247\cdot4)^2+(239\cdot5)^2+(209\cdot2)^2}{5} - 53{,}296\cdot3333$$
$$= \frac{267{,}400\cdot49}{5} - 53{,}296\cdot3333 = 53{,}480\cdot0980 - 53{,}296\cdot3333$$
$$= 183\cdot7647.$$

$$\text{Treatment } SS = \frac{(242\cdot4)^2+(234\cdot7)^2+(205\cdot2)^2+(215\cdot0)^2+(257\cdot0)^2}{5} - 53{,}296\cdot3333$$
$$= \frac{268{,}222\cdot89}{5} - 53{,}296\cdot3333 = 53{,}644\cdot5780 - 53{,}296\cdot3333$$
$$= 348\cdot2447.$$

Error SS = Total SS − Row SS − Column SS − Treatment SS
$= 304\cdot0826.$

TABLE 2.5

ANALYSIS OF VARIANCE TABLE FOR THE *LSD*

Source of variation	df	SS	MS	F_0
Rows	4	141·0847	35·2712	
Columns	4	183·7647	45·9412	
Treatments	4	348·2447	87·0612	3·436
Error	12	304·0826	25·3402	
Total	24	977·1767		—

As $F_{.01\,;\,4,12} = 5\cdot41$ and $F_{.05\,;\,4,12} = 3\cdot26$, the hypothesis of no treatment effect is accepted at the 1% level but is rejected at the 5% level.

6·13. BALANCED INCOMPLETE BLOCK DESIGNS

The precision of the estimate of a treatment effect depends on the number of replications of the treatment *i.e.*, the larger is the number of replications, the more is the precision. A similar thing holds for the precision of the estimate of the difference between two treatment effects. If two treatments occur together in a block, then we can say that these are replicated once in that block. Similarly, if there are say, p blocks in a design in each of which the two treatments occur together, then the pair of treatments is said to be replicated p times in the design. The precision of the estimate of the difference between two treatments depends on the number of replications of the two treatments.

If in a block the number of experimental units or plots is smaller than the number of treatments, then the block is said to be incomplete and a design constituted of such blocks is called an '*incomplete block design*'.

As the name suggests, the balanced incomplete block designs are arranged in blocks or groups that are smaller than a complete replication in order to eliminate heterogeneity to a greater extent than is possible with randomised block design and Latin square design. These designs were introduced by ***F. Yates*** in a paper "A new method of arranging variety trials involving a large number of varieties" Journal Agr. Sci. 26, 424-455, 1936.

In factorial experiments confounding enables us to reduce the size of the block at the cost of information on certain treatment comparisons which may be relatively of less importance. But in Balanced Incomplete block designs (BIBD) which were developed for experiments in plant breeding and agriculture selection of all comparisons among pairs of treatments is made with equal precision.

Incomplete Block Design (I.B.D.). ***Definition.*** *An incomplete block design is one having v treatments and b blocks each of size k such that each of the treatments is replicated r times and each pair of treatments occurs once and only once in the same block.* v, b, r and k are known as the parameters of the I.B.D.

Balanced Incomplete Block Design (BIBD). ***Definition.*** *An arrangement of v treatments in b blocks of k plots each ($k < v$) is known as BIBD, if*

(*i*) *each treatment occurs once and only once in r blocks and*

(*ii*) *each pair of treatments occurs together in λ blocks.*

BIBD is used when all treatment comparisons are equally important as it ensures equal precisions of the estimates of all pairs of treatment effects. [Apart from ensuring equal precisions all the treatments, the variance of the estimate of any treatment variety mean, is σ^2/r, where σ^2 is the error variance and r the number of replicates, which is same for each treatment.]

6·13·1. Parameters of B.I.B.D. The integers v, r, b, k and λ are called the parameters of the B.I.B.D., where

v = number of varieties or treatments, b = number of blocks

k = block size, r = number of replicates for each treatment

λ = number of blocks in which any pair of treatments occurs together or number of times any two treatments occur together in a block. The following parametric relations serve as a necessary condition for the existence of a B.I.B.D.

(*i*) $vr = bk$, (*ii*) $\lambda(v-1) = r(k-1)$, and (*iii*) $b \geq v$ (Fisher's Inequality.)

We shall now establish these results as theorems 6·1 to 6·3 on B.I.B.D.

Theorem 6·1.

$$vr = bk \qquad \ldots(6\cdot261)$$

Proof. Since there are v treatments each replicated r times, total number of plots in the design is vr. Further since there are b blocks each of size k, there are bk plots in all.

Hence,

$$vr = bk$$

6·13·2. Incidence Matrix. Associated with any design D is the *incidence matrix* $N = (n_{ij})$, $(i = 1, 2, \ldots, v\,; j = 1, 2, \ldots, b)$, where n_{ij} denotes the number of times the ith treatment occurs in the jth block. Thus by the definition of a BIBD,

$$N = \begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ n_{21} & n_{22} & \cdots & n_{2b} \\ \vdots & \vdots & \vdots & \vdots \\ n_{v1} & n_{v2} & \cdots\cdots & n_{vb} \end{bmatrix} \qquad \ldots(6\cdot262)$$

where $n_{ij} = 1$, if ith treatment occurs in the jth block. $\ldots(6\cdot262a)$
$\qquad = 0$, otherwise

Remark. Since in case of BIBD, n_{ij} can take only two values 0 or 1, BIBD is sometimes called a *binary design*.

We also observe, by definition of BIBD :

$$\sum_{j=1}^{b} n_{ij} = \sum_{j=1}^{b} n_{ij}^2 = r\,;\ (i = 1, 2, \ldots, v) \qquad \ldots(6\cdot263)$$

$$\sum_{i=1}^{v} n_{ij} = \sum_{i=1}^{v} n_{ij}^2 = k\,;\ (j = 1, 2, \ldots, b) \qquad \ldots(6\cdot263a)$$

$$\sum_{j=1}^{b} n_{ij}\, n_{lj} = \lambda\,;\ (i \neq l = 1, 2, \ldots, v), \qquad \ldots(6\cdot263b)$$

since $n_{ij}\, n_{lj} = 1$ if and only if ith and lth treatments occur together in the jth block otherwise it is zero and they occur together in λ blocks.

If N' denotes the transpose of N then

$$NN' = \begin{bmatrix} \sum_j n_{1j}^2 & \sum_j n_{1j}\, n_{2j} & \cdots & \sum_j n_{1j}\, n_{vj} \\ \sum_j n_{2j}\, n_{1j} & \sum_j n_{2j}^2 & \cdots & \sum_j n_{2j}\, n_{vj} \\ \vdots & \vdots & & \vdots \\ \sum_j n_{vj}\, n_{1j} & \sum_j n_{vj}\, n_{2j} & \cdots & \sum_j n_{vj}^2 \end{bmatrix} = \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda & \lambda & \lambda & \cdots & r \end{bmatrix}_{v \times v} \qquad \ldots(6\cdot264)$$

[From 6·263) and (6·263b)]

Theorem 6·2.

$$\lambda(v-1) = r\,(k-1). \qquad \ldots(6\cdot265)$$

Proof. Let us denote by E_{mn} the $m \times n$ matrix all of whose elements are unity. From (6·264), we get

$$NN'\, E_{v1} = \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda & \lambda & \lambda & \cdots & r \end{bmatrix}_{v \times v} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{v \times 1} = \left[r + \lambda\,(v-1)\right] \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= \left[r + \lambda\,(v-1)\right] E_{v1} \qquad \ldots(6\cdot266)$$

Also

$$\mathbf{NN'}E_{v1} = \mathbf{N}(\mathbf{N'}E_{v1})$$

$$= N\begin{bmatrix} n_{11} & n_{21} & \cdots & n_{v1} \\ n_{12} & n_{22} & \cdots & n_{v2} \\ \vdots & \vdots & & \vdots \\ n_{1b} & n_{2b} & \cdots & n_{vb} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = N\begin{bmatrix} \sum_i n_{i1} \\ \sum_i n_{i2} \\ \vdots \\ \sum_i n_{ib} \end{bmatrix} = N\begin{bmatrix} k \\ k \\ \vdots \\ k \end{bmatrix} \quad \text{[From (6·263}a\text{)]}$$

$$= k\begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ n_{21} & n_{22} & \cdots & n_{2b} \\ \vdots & \vdots & & \vdots \\ n_{v1} & n_{v2} & \cdots & n_{vb} \end{bmatrix}_{v\times b}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{b\times 1} = k\begin{bmatrix} \sum_j n_{1j} \\ \sum_j n_{2j} \\ \vdots \\ \sum n_{vj} \end{bmatrix} = k\begin{bmatrix} r \\ r \\ \vdots \\ r \end{bmatrix}_{v\times 1} \quad \text{[From (6·263)]}$$

$$= kr\,E_{v1} \qquad \ldots (6·266a)$$

From (6·266) and (6·266a), we get

$$[r + \lambda(v-1)]\,E_{v1} = kr\,E_{v1} \quad \Rightarrow \quad r + \lambda(v-1) = kr \quad i.e., \quad \lambda(v-1) = r(k-1), \text{ as desired.}$$

Aliter. v treatments gives rise to ${}^vC_2$ pairs and since each pair occurs λ times, the total number of times all the pairs occur in the design is $\lambda\,{}^vC_2$

Further since the size of each block is k, each block gives rise to ${}^kC_2$ pairs and since there are b blocks in all, the total number of treatment pairs in all the blocks is $b\,{}^kC_2$. Hence, we get

$$\lambda\,{}^vC_2 = b\,{}^kC_2 \quad \Rightarrow \quad \lambda v(v-1) = bk(k-1) \quad i.e., \quad \lambda(v-1) = \frac{bk}{v}(k-1)$$

$$\therefore \quad \lambda(v-1) = r(k-1), \text{ as desired.} \qquad [\because\ vr = bk]$$

Theorem 6·3. $b \geq v$ (*Fisher's Inequality*).

Proof. From (6·84), the determinant of the matrix **N N′** is given by

$$|\mathbf{N\,N'}| = \begin{vmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda & \lambda & \lambda & & r \end{vmatrix}_{v\times v}$$

Adding 2nd, 3rd, ..., vth columns to the first column and taking $[r + (v-1)\lambda]$ common from the first column, we get

$$\mathbf{N\,N'} = [r + (v-1)\lambda]\begin{vmatrix} 1 & \lambda & \lambda & \cdots & \lambda \\ 1 & r & \lambda & \cdots & \lambda \\ 1 & \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda & \lambda & \cdots & r \end{vmatrix} = [r + (v-1)\lambda]\begin{vmatrix} 1 & \lambda & \lambda & \cdots & \lambda \\ 0 & (r-\lambda) & 0 & \cdots & 0 \\ 0 & 0 & (r-\lambda) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & (r-\lambda) \end{vmatrix}$$

(Subtracting first row from the 2nd, 3rd, ..., vth row)

$$= [r + (v-1)\lambda]\,(r-\lambda)^{v-1} \qquad \text{(Expanding by first column)}$$

$$= rk\,(r-\lambda)^{v-1} \qquad \text{[Using (6·265)]} \qquad \ldots(6·268a)$$

Thus $| \mathbf{N N'} | \neq 0$, for if $r = \lambda$ then from (6·265), we get

$$(v - 1) = (k - 1) \quad \Rightarrow \quad v = k$$

indicating that the design reduces to randomised block design. Hence, $\mathbf{N N'}$ is non-singular and consequently

$$\text{Rank } (\mathbf{N N'}) = v, \quad \text{...(6·269)}$$

since v is the order of the matrix $\mathbf{N N'}$.

But

$$\text{Rank } (\mathbf{N N'}) = \text{Rank } (\mathbf{N})$$

$\therefore$

$$\text{Rank } (\mathbf{N}) = v \quad \text{[From (6·269)]} \quad \text{...(6·270)}$$

But since $\mathbf{N}$ is a $v \times b$ matrix, its rank can be at most b.

$$v = \text{rank } \mathbf{N} \leq b \quad \Rightarrow \quad b \geq v, \quad \text{as desired.} \quad \text{...(6·270}a\text{)}$$

Deductions. (*i*)

$$r \geq k$$

Proof. We have

$$vr = bk \quad \Rightarrow \quad r = \frac{b}{v} \cdot k$$

Since

$$b \geq v, \text{ we get } \; r \geq k \quad \text{...(6·272)}$$

(*ii*)

$$b \geq v + r - k \quad \text{[From (6·271)]}$$

Proof. We have $v - k \geq 0$ and $r - k \geq 0$

$$\therefore \quad (v - k)(r - k) \geq 0 \quad \Rightarrow \quad \left(\frac{v}{k} - 1\right)(r - k) \geq 0 \quad i.e., \quad \frac{v}{k}(r - k) - (r - k) \geq 0$$

$$\therefore \quad \frac{vr}{k} - v \geq r - k \quad \Rightarrow \quad b \geq v + r - k \quad [\because \; vr = bk]$$

6·13·3. Symmetric BIBD

Definition. *A BIBD is said to be symmetric if $b = v$ and $r = k$.*

In this case the incidence matrix $\mathbf{N}$ is a square matrix.

Substituting $r = k$ and $b = v$ in (6·88*a*), we get

$$| \mathbf{N N'} | = r^2 (r - \lambda)^{v-1} \Rightarrow | \mathbf{N} | \, | \mathbf{N'} | = r^2 (r - \lambda)^{v-1} \; i.e., \; | \mathbf{N} |^2 = r^2 (r - \lambda)^{v-1} \quad \text{...(6·273)}$$

$$\therefore \quad | \mathbf{N} | = \pm r (r - \lambda)^{\frac{v-1}{2}} \quad [\because \; | \mathbf{N} | = | \mathbf{N'} |]^{v} \quad \text{...(6·274)}$$

Since the determinant of the incidence matrix $\mathbf{N}$ is an integer, hence when v is even, $(r - \lambda)$ must be a perfect square.

Remarks **1.** Necessary condition for a symmetrical BIBD with v as even is that $(r - \lambda)$ must be a perfect square.

2. If $\mathbf{N}$ is the incidence matrix of BIBD, then

(*i*) every row sum is r,

(*ii*) every column sum is k,

(*iii*) the inner product of any two rows of $\mathbf{N}$ is λ.

For example, let us consider the following BIBD with $v = 4$, $b = 6$, $k = 2$, $r = 3$, $\lambda = 1$

BIBD Treatments

Blocks		
1	1	2
2	1	3
3	1	4
4	2	3
5	2	4
6	3	4

$$N_{4\times 6} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

We note that everyrow sum in **N** is $3 = r$, every column sum is $2 = k$ and the inner product of any two rows, *e.g.*, say 2nd and 3rd rows is

$$[1\ 0\ 0\ 1\ 1\ 0] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 1 = \lambda$$

Theorem 6·4. *In a symmetric BIBD, the number of treatments common between any two blocks is* λ.

Proof. We have already proved in (6·84) that

$$\mathbf{N\,N'} = \begin{bmatrix} r & \lambda & \lambda & \dots & \lambda \\ \lambda & r & \lambda & \dots & \lambda \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda & \lambda & \lambda & \dots & r \end{bmatrix}_{v\times v} = \begin{bmatrix} r-\lambda & 0 & 0 & \dots & 0 \\ 0 & r-\lambda & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & r-\lambda \end{bmatrix} + \begin{bmatrix} \lambda & \lambda & \dots & \lambda \\ \lambda & \lambda & \dots & \lambda \\ \vdots & \vdots & & \vdots \\ \lambda & \lambda & \dots & \lambda \end{bmatrix}$$

$$= (r-\lambda)\,\mathbf{I}_v + \lambda\,\mathbf{E}_{vv} \qquad \dots(6\cdot275)$$

where $\mathbf{I}_v$ is a unit matrix of order v.

Also for a symmetric BIBD, we have (for proof, see remark below) :

$$(\mathbf{N\,N'})^{-1} = \frac{1}{r-\lambda}\left[\mathbf{I}_v - \frac{\lambda}{r^2}\mathbf{E}_{vv}\right] \quad\Rightarrow\quad (\mathbf{N'})^{-1}\,\mathbf{N}^{-1} = \frac{1}{r-\lambda}\left[\mathbf{I}_v - \frac{\lambda}{r^2}\mathbf{E}_{vv}\right] \qquad \dots(6\cdot276)$$

Premultiplying by $(\mathbf{N'})$, we get $\quad \mathbf{N}^{-1} = \dfrac{1}{r-\lambda}\left[\mathbf{N'} - \dfrac{\lambda}{r^2}\mathbf{N'\,E}_{vv}\right] \qquad \dots(6\cdot277)$

But it can be easily verified that for symmetric BIBD

$$\mathbf{N'\,E}_{vv} = \mathbf{N\,E}_{vv} = r\,\mathbf{E}_{vv} = k\,\mathbf{E}_{vv} \quad\Rightarrow\quad \frac{\mathbf{N'}}{r}\mathbf{E}_{vv} = \mathbf{E}_{vv} \qquad (\because\ r = k) \qquad \dots(6\cdot278)$$

Substituting in (6·277), we get $\quad \mathbf{N}^{-1} = \dfrac{1}{r-\lambda}\left[\mathbf{N'} - \dfrac{\lambda}{r}\,\mathbf{E}_{vv}\right]$

Post multiplying by **N**, we have

$$\mathbf{I}_v = \frac{1}{r-\lambda}\left[\mathbf{N'\,N} - \frac{\lambda}{r}\mathbf{N\,E}_{vv}\right] = \frac{1}{r-\lambda}\left[\mathbf{N'\,N} - \lambda\,\mathbf{E}_{vv}\right] \qquad \text{[From (6·278)]}$$

$$\Rightarrow \quad \mathbf{N'\,N} = (r-\lambda)\,\mathbf{I}_v + \lambda\,\mathbf{E}_{vv} \qquad \dots(6\cdot279)$$

From (6·275) and (6·279), we get for a symmetric BIBD

$$\sout{N'N = N'N} \quad NN' = N'N. \qquad \ldots(6{\cdot}280)$$

Thus, the inner product of any two rows of **N** is equal to the inner product of any two columns of **N**, *i.e.*, λ.

Hence, in case of a symmetric BIBD, any two blocks have λ treatments in common.

Remark.

$$(NN')^{-1} = \frac{1}{|NN'|} \text{Adj}(NN') \qquad \ldots(1)$$

The cofactor of diagonal element of **N N′** $= \begin{vmatrix} r & \lambda & \ldots & \lambda \\ \lambda & r & \ldots & \lambda \\ \vdots & \vdots & & \vdots \\ \lambda & \lambda & \ldots & r \end{vmatrix}_{(v-1)\times(v-1)}$

$= (r-\lambda)^{v-2}\,[r + (v-2)\lambda] = (r-\lambda)^{v-2}\,[r + (v-1)\lambda - \lambda]$ [*c.f.* (6·269)]

$= r - \lambda)^{v-2}\,[rk - \lambda]$ [$\because (v-1)\lambda = r(k-1)$]

$= r^2(r-\lambda)^{v-2} - \lambda(r-\lambda)^{v-2} = A$, (say). ($\because$ $r = k$ for symmetric BIBD) ...(2)

The co-factor of off-diagonal element of **N N′**

$$= -\begin{vmatrix} \lambda & \lambda & \ldots & \lambda \\ \lambda & r & \ldots & \lambda \\ \vdots & \vdots & & \vdots \\ \lambda & \lambda & \ldots & r \end{vmatrix}_{(v-1)\times(v-1)} = -\begin{vmatrix} \lambda & 0 & \ldots & 0 \\ \lambda & (r-\lambda) & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \lambda & 0 & \ldots & (r-\lambda) \end{vmatrix}_{(v-1)\times(v-1)}$$

$= -\lambda(r-\lambda)^{v-2} = B$, (say). ... (3)

Since **N N′** is symmetric *i.e.*, **N N′ = N′ N**, the transpose of the matrix of cofactors of elements of **N N′** is same as the matrix of cofactors of **N N′**,

Also, for a symmetric BIBD,

$|NN'| = r^2(r-\lambda)^{v-1} = C$, (say). [From (6·273)] ...(4)

Substituting from (2), (3) and (4) in (1), we get

$$(NN')^{-1} = \begin{bmatrix} A/C & B/C & \ldots & B/C \\ B/C & A/C & \ldots & B/C \\ \vdots & \vdots & & \vdots \\ B/C & B/C & \ldots & A/C \end{bmatrix}_{v\times v} = \frac{1}{(r-\lambda)} \begin{bmatrix} 1-\frac{\lambda}{r^2} & -\frac{\lambda}{r^2} & \ldots & -\frac{\lambda}{r^2} \\ -\frac{\lambda}{r^2} & 1-\frac{\lambda}{r^2} & \ldots & -\frac{\lambda}{r^2} \\ \vdots & \vdots & & \vdots \\ -\frac{\lambda}{r^2} & -\frac{\lambda}{r^2} & \ldots & 1-\frac{\lambda}{r^2} \end{bmatrix}_{v\times v}$$

$$= \frac{1}{(r-\lambda)} \left\{ \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} + \begin{bmatrix} -\frac{\lambda}{r^2} & -\frac{\lambda}{r^2} & \ldots & -\frac{\lambda}{r^2} \\ -\frac{\lambda}{r^2} & -\frac{\lambda}{r^2} & \ldots & -\frac{\lambda}{r^2} \\ \vdots & \vdots & & \vdots \\ -\frac{\lambda}{r^2} & -\frac{\lambda}{r^2} & \ldots & -\frac{\lambda}{r^2} \end{bmatrix} \right\}$$

$$\Rightarrow \quad (\mathbf{N}\,\mathbf{N}')^{-1} = \frac{1}{r-\lambda}\left(\mathbf{I}_v - \frac{\lambda}{r}\mathbf{E}_{vv}\right) \qquad \ldots(6\cdot 281)$$

***Theorem 6·5.** If N is the incidence matrix of a symmetric BIBD, then*

$$(\mathbf{N}\,\mathbf{N}')\,(\mathbf{N}'\,\mathbf{N}) = (r-\lambda)\,\mathbf{N}'\,\mathbf{N} + k^2\,\lambda\,\mathbf{E}_{vv} \qquad \ldots(6\cdot 282)$$

Proof. For a symmetric BIBD, we have

$$\mathbf{N}'\,\mathbf{N} = (r-\lambda)\,\mathbf{I}_v + \lambda\mathbf{E}_{vv} \qquad \text{[From (6·279)]}$$

$$\Rightarrow \quad (\mathbf{N}'\,\mathbf{N})\,(\mathbf{N}'\,\mathbf{N}) = \mathbf{N}'\,\mathbf{N}\,[(r-\lambda)\,\mathbf{I}_v + \lambda\mathbf{E}_{vv}] = (r-\lambda)\,\mathbf{N}'\,\mathbf{N} + \lambda(\mathbf{N}'\,\mathbf{N})\,\mathbf{E}_{vv}$$

$$= (r-\lambda)\,\mathbf{N}'\,\mathbf{N} + \lambda\mathbf{N}'\,(\mathbf{N}\,\mathbf{E}_{vv}) = (r-\lambda)\,\mathbf{N}'\,\mathbf{N} + \lambda\mathbf{N}'\,(k\,\mathbf{E}_{vv}) \qquad \text{[From (6·278)]}$$

$$= (r-\lambda)\,\mathbf{N}'\,\mathbf{N} + \lambda k\,(\mathbf{N}'\,\mathbf{E}_{vv}) = (r-\lambda)\,\mathbf{N}'\,\mathbf{N} + \lambda k.k\,\mathbf{E}_{vv} \qquad \text{[From (6·278)]}$$

$$= (r-\lambda)\,\mathbf{N}'\,\mathbf{N} + \lambda k^2\,\mathbf{E}_{vv}$$

6·13·4. Resolvable Design. ***Definition.*** *A BIBD with parameters v, r, b, k and λ is said to be resolvable if the b blocks can be divided into r groups or sets of b/r blocks each, b/r being an integer, such that b/r blocks forming any of these sets give a complete replication of all the v treatements.*

TABLE 6·76 : BIBD

Blocks	Treatments		
1	1	2	First set
2	3	4	
3	1	3	Second set
4	2	4	
5	1	4	Third set
6	2	3	

For example, let us consider the BIBD with parameters $v = 4$, $b = 6$, $r = 3$, $k = 2$ and $\lambda = 1$, as given in *Table 6·76*.

Here $b = 6$ blocks are divided into $r = 3$ sets each of $b/r = 6/3 = 2$ (integer) blocks. Moreover, each set contains each of the treatments occurring once and only once. Also $\lambda = 1$.

Hence, the above design is resolvable BIBD.

***Theorem 6·6.** For a resolvable BIBD with parameters v, b, r, k, λ*

$$b \geq v + r - 1 \qquad \ldots(6\cdot 283)$$

Proof. Since the design is resolvable, b/r must be an integer, equal to n (say), *i.e.*,

$$b/r = n \quad \Rightarrow \quad b = rn \qquad \ldots(6\cdot 284)$$

But for a BIBD

$$vr = bk \quad \Rightarrow \quad vr = rnk \quad \text{i.e.,} \quad v = nk \qquad \ldots(6\cdot 284a)$$

Also for a BIBD

$$r(k-1) = \lambda\,(v-1) = \lambda(nk-1) \qquad \text{[From (6·284a)]}$$

$$\Rightarrow \quad r = \frac{\lambda\,(nk-1)}{k-1} = \frac{\lambda n\,(k-1) + \lambda n - \lambda}{k-1} = \lambda n + \frac{\lambda\,(n-1)}{(k-1)}$$

$$\therefore \quad r - \lambda n = \frac{\lambda\,(n-1)}{(k-1)} \qquad \ldots(6\cdot 285)$$

Since r is an integer, and λ and n are also integers, from (6·285), we conclude that $\lambda(n-1)/(k-1)$ must be an integer.

Now, if possible let

$$b < v + r - 1), \quad i.e., \quad b - r < r - 1 \qquad \ldots(*)$$

$$r(n-1) < (v-1) \qquad \text{[From (6·284)]}$$

$$\Rightarrow \quad r(n-1) < \frac{r(k-1)}{\lambda} \qquad [\because r(k-1) = \lambda(v-1)]$$

$$\Rightarrow \quad \frac{\lambda(n-1)}{k-1} < 1,$$

$\Rightarrow$

which is a contradiction of the fact that $\lambda(n-1)/(k-1)$ is an integer. Hence, our assumption (*) is wrong and we must have

$$b \geq v + r - 1$$

Aliter. The incidence matrix N of the design consists of r sets of b/r rows each, where any set of rows is such that it occurs once and only once in each column of the set. By adding the 1st, 2nd, ..., $\{(b/r) - 1\}$th row, to the (b/r) th row of a set we obtain a row consisting of ones only. Moreover, since there are r sets, for each of these sets the rows add up to the same vector (1, 1, ..., 1). We know that if any elementary row transformation makes r row vectors identical then the rank is at least reduced by $(r-1)$. Hence using (6·270), we have

$$v = \text{Rank } \mathbf{N} \leq b - (r-1) \quad \Rightarrow \quad b \geq v + r - 1$$

6·13·5. Affine Resolvable Design.

Definition. *A resolvable design is said to be affine resolvable if $b = r + v - 1$ and any two blocks from different sets have k^2/v treatments common where k^2/v is an integer.*

For example, let us consider the resolvable design of § 6·10·3 with parameters $v = 4$, $b = 6, r = 3, k = 2$ and $\lambda = 1$.

We observe that the condition $b = v + r - 1$ is satisfied. Also $(k^2/v) = 1$ (integer) and any two blocks from different sets have only one treatment $(\lambda = 1)$ common. Hence, the design is *affine resolvable*.

6·13·6. Analysis of Balanced Incomplete Block Design. ***(Intra Block Analysis of BIBD).*** This method of analysis developed by F. Yates by the use of standard Least Square Technique is sometimes referred to as the *'intra-block'* analysis, *i.e.,* the analysis without recovery of interblock information.

Consider 'a' units of material comprising b blocks to which v treatments are applied such that each treatment is replicated r times subject to the conditions of the BIBD.

Let $\mathbf{N} = (n_{ij})$ be the incidence matrix of the BIBD, where

$n_{ij} = 1$, if ith treatment occurs in the jth block

$= 0$, otherwise

Then as given in (6·263), (6·263a) and (6·263b), we have

$$\sum_{j=1}^{b} n_{ij} = \sum_{j=1}^{b} n_{ij}^2 = r \quad ; \quad \sum_{i=1}^{v} n_{ij} = \sum_{i=1}^{v} n_{ij}^2 = k \quad \text{and} \quad \sum_{j=1}^{b} n_{ij}\, n_{lj} = \lambda, (i \neq l) \qquad \ldots(6·286)$$

Let y_{ij} $(i = 1, 2, ..., v;\ j = 1, 2, ..., b)$ be the observation recorded on a unit of the material for the ith treatment which we suppose to be in the jth block where (i, j) runs through the set D for which $n_{ij} = 1$. Here the mathematical model is :

$$y_{ij} = \mu + t_i + b_j + \varepsilon_{ij} \text{ for } (i, j) \in D \qquad \ldots(6·287)$$

where μ is the general mean effect, t_i ($i = 1, 2, ..., v$) is the effect of the ith treatment, b_j ($j = 1, 2, ..., b$) is the effect of the jth block and ε_{ij} are the $v \times r$ intra-block errors which are assumed to be independently normally distributed with mean zero and common variance σ_e^2, *i.e.*, ε_{ij} are *i.i.d.* $N(0, \sigma_e^2)$. *In the intra block analysis we assume that the treatment effects and block effects are fixed, though unknown.*

According to the principle of least squares, the normal equations for estimating the $(v + b + 1)$ parameters t_i, ($i = 1, 2, ..., v$) ; b_j, ($j = 1, 2, ..., b$) and μ are obtained on minimising the error or residual sum of squares.

$$E = \sum_{(i, j) \in D} \varepsilon_{ij}^2 = \sum_{(i, j) \in D} (y_{ij} - \mu - t_i - b_i)^2$$

Equating to zero the partial derivatives of E *w.r.t.*, μ, t_i and b_j, we get the normal equations as :

$$\frac{\partial E}{\partial \mu} = -2 \sum_{(i, j) \in D} (y_{ij} - \hat{\mu} - \tau_i - \beta_j) = 0 \quad ...(6·287a)$$

$$\frac{\partial E}{\partial t_i} = -2 \sum_{j \in D_i} (y_{ij} - \hat{\mu} - \tau_i - \beta_j) = 0 \quad ...(6·287b)$$

$$\frac{\partial E}{\partial b_i} = -2 \sum_{i \in D_i} (y_{ij} - \hat{\mu} - \tau_i - \beta_j) = 0 \quad ...(6·287c)$$

where μ, τ_i and β_j are the least square estimates of μ, t_i and b_j respectively, D_i is the set of r values for which $(i, j) \in D$ for given i and D_j is the set of k values for which $(i, j) \in D$ for given j.

In order that the set of equations (6·287a), (6·287b) and (6·287c) has an unique solution, we must impose the restrictions

$$\sum_{i=1}^{v} \tau_i = 0, \qquad \sum_{j=1}^{b} \beta_j = 0 \quad ...(6·288)$$

(6·287a) gives : $\displaystyle\sum_{(i, j) \in D} y_{ij} - rv\hat{\mu} - \sum_{(i, j) \in D} \tau_i - \sum_{(i, j) \in D} \beta_j = 0$ ...(6·289)

Now $\displaystyle\sum_{(i, j) \in D} \tau_i = \sum_i \left(\sum_{j \in D_i} \tau_i \right) = \sum_i (r\tau_i) = r \sum_i \tau_i = 0$ [From (6·288)] ...(6·290)

Similarly, $\displaystyle\sum_{(i, j) \in D} \beta_j = \sum_j \left(\sum_{j \in D_j} \beta_j \right) = k \sum_j \beta_j = 0$ [From (6·288)] ...(6·290a)

Substituting in (6·289), we get

$$\hat{\mu} = \frac{\sum_{(i, j) \in D} y_{ij}}{rv} = \bar{y}.. \quad ...(6·291)$$

From (6·105b), we get

$$\sum_{j \in D_i} y_{ij} - r\hat{\mu} - r\tau_i - \sum_{j \in D_i} \beta_j = 0 \quad \Rightarrow \quad T_i = r\,y.. + r\tau_i + \sum_{j \in D_i} \beta_j$$

where $T_i = \sum_{j \in D_i} y_{ij}$ is the total yield for the ith treatment. Further, since D_i is the set of r values for which $(i, j) \in D$ for given i, *i.e.*, $n_{ij} = 1$ and i has a fixed value, we write

$$T_i = r\,\bar{y}_{..} + r\,\tau_i + \sum_{j=1}^{b} n_{ij}\,\beta_j \qquad \ldots(6\cdot292)$$

From (6·292) we observe that the estimate of a treatment effect is no longer of the form of the observed treatment mean minus the grandmean, since the block effects do not enter in the same way into all the observed treatment means. For example, a treatment may be favoured by occurring only in blocks with high block effects.

Similarly from (6·287*c*), we shall get

$$B_j = k\,\bar{y}_{..} + k\beta_j + \sum_{i=1}^{v} n_{ij}\,\tau_i \qquad \ldots(6\cdot293)$$

where $B_j = \sum_{i \in D_i} y_{ij}$ is the total yield from the jth block.

The quantity :
$$\frac{B_j}{k} - \bar{y}_{..} = \beta_j + \frac{1}{k}\sum_{i=1}^{v} n_{ij}\,\tau_i \qquad \ldots(6\cdot294)$$

may be called the estimate of the jth block effect ignoring treatments.

Substituting the value of β_j from (6·293) in (6·292), we get

$$T_i = r\,\bar{y}_{..} + r\,\tau_i + \frac{1}{k}\sum_{j=1}^{b}\left[n_{ij}\left(B_j - k\,\bar{y}_{..} - \sum_{p=1}^{v} n_{pj}\,\tau_p\right)\right]$$

$$\Rightarrow \quad T_i - \frac{1}{k}\sum_{j=1}^{b} n_{ij}\,B_j = r\,\bar{y}_{..} + r\tau_i - \bar{y}_{..}\sum_{j=1}^{b} n_{ij} - \frac{1}{k}\sum_{j=1}^{b}\sum_{p=1}^{v} n_{ij}\,n_{pj}\,\tau_p$$

$$= r\tau_i - \frac{1}{k}\sum_{j=1}^{b}\sum_{p=1}^{v} n_{ij}\,n_{pj}\,\tau_p \qquad \ldots(6\cdot295)$$

The quantity :
$$Q_i = T_i - \sum_{j=1}^{b}\frac{n_{ij}\,B_j}{k} \qquad \ldots(6\cdot296)$$

is called the ith adjusted treatment total or the adjusted total yield for the ith treatment. The adjustment consists in subtracting from the treatment total T_i the sum of the jth block average B_j/k (average yield per plot for the jth block) for the blocks in which the ith treatment occurs.

Thus from (6·294) and (6·295), we have

$$Q_i = r\,\tau_i - \frac{1}{k}\sum_{j=1}^{b}\sum_{p=1}^{v} n_{ij}\,n_{pj}\,\tau_p$$

$$= -\frac{\tau_1}{k}\sum_{j=1}^{b} n_{ij}\,n_{1j} - \frac{\tau_2}{k}\sum_{j=1}^{b} n_{ij}\,n_{2j} - \ldots + \tau_i\left(r - \frac{1}{k}\sum_{j=1}^{b} n_{ij}^2\right) - \ldots - \frac{\tau_v}{k}\sum_{j=1}^{b} n_{ij}\,n_{vj}$$

$$= -\frac{\lambda}{k}\left(\tau_1 + \tau_2 + \ldots + \tau_v\right) + \left(r - \frac{r}{k} + \frac{\lambda}{k}\right)\tau_i \qquad \text{[Using (6·286)]}$$

$$= \left(r - \frac{r}{k} + \frac{\lambda}{k}\right)\tau_i \qquad \left(\because \sum_{i=1}^{v} \tau_i = 0\right)$$

$$= \left[\frac{r(k-1)+\lambda}{k}\right]\tau_i = \left[\frac{\lambda(v-1)+\lambda}{k}\right]\tau_i \qquad \text{[From (6·265)]}$$

$$Q_i = \frac{\lambda v}{k}\tau_i \quad \Rightarrow \quad \tau_i = Q_i - \frac{k}{\lambda v} \; ; \; (i = 1, 2, \ldots, v) \qquad \ldots(6\cdot297)$$

$\therefore$

If we write :

$$E = \frac{\lambda v}{kr} \qquad \ldots(6\cdot297a)$$

Then,

$$\tau_i = \frac{Q_i}{rE} \qquad \ldots(6\cdot297b)$$

It may be remarked that $E < 1$, since

$$E = \frac{v}{k}\cdot\frac{\lambda}{r} = \frac{v(k-1)}{k(v-1)} = \frac{vk - v}{vk - k} < 1 \qquad [\because \; \lambda(v-1) = r(k-1) \text{ and } v > k] \qquad \ldots(6\cdot298)$$

The quantity $\left(\sum_{i=1}^{v} \tau_i Q_i\right)$ is the sum of squares due to treatments after adjusting for block effects. Thus

$$\text{S.S. Treatments (adjusted)} = \sum_{i=1}^{v} \tau_i Q_i = \sum_{i=1}^{v} \frac{Q_i^2}{rE} \qquad \ldots(6\cdot299)$$

The quantity

$$\sum_{j=1}^{b} (B_j^2/k) - n\hat{\mu}\bar{y}_{..} = \sum (B_j^2/k) - \frac{1}{n}\left[\sum_{(i,j)\in D} y_{ij}\right]^2 = \sum_{j=1}^{b} (B_j^2/k) - \text{C.F.} \qquad \ldots(6\cdot300)$$

Under the model (6·287), the residual sum of squares is given by :

$$\sum_{(i,j)\in D} (y_{ij} - \hat{\mu} - \tau_i - \beta_j)^2 = \sum_{(i,j)\in D} y_{ij}^2 - n\hat{\mu}\bar{y}_{..} - \sum_{i=1}^{v} T_i\tau_i - \sum_{j=1}^{b} B_j\beta_j$$

The sum of the squares due to $\hat{\mu}$, τ_i and β_j being given by ($n = vr = bk$)

$$S(\mu, b, \tau) = n\hat{\mu}\bar{y}_{..} + \sum_{i=1}^{v} T_i\tau_i + \sum_{j=1}^{b} B_j\beta_j = \hat{\mu}\sum_{(i,j)\in D} y_{ij} + \sum_{j=1}^{b} B_j\beta_j + \sum_{i=1}^{v} T_i\tau_i$$

$$= \hat{\mu}\sum_j\left(\sum_{i\in D_i} y_{ij}\right) + \sum_j B_j\beta_j + \sum_i T_i\tau_i = \hat{\mu}\sum_j B_j + \sum_j B_j\beta_j + \sum_i T_i\tau_i$$

$$= \sum_j (\hat{\mu} + \beta_j) B_j + \sum_i T_i\tau_i$$

Substituting for $\hat{\mu} + \beta_j$ from (6·293), we get

$$S(\mu, b, \tau) = \sum_{j=1}^{b}\left[\frac{1}{k}\left(B_j - \sum_{i=1}^{v} n_{ij}\tau_i\right)B_j\right] + \sum_{i=1}^{v} T_i\tau_i$$

$$= \sum_{j=1}^{b} (B_j^2/k) + \sum_{i=1}^{v} \tau_i\left(T_i - \sum_j n_{ij}B_j/k\right) = \sum_{j=1}^{b} (B_j^2/k) + \sum_{i=1}^{v} \tau_i Q_i$$

$$= n\hat{\mu}\bar{y}_{..} + \left(\sum_{j=1}^{b} \frac{B_j^2}{k} - n\hat{\mu}\bar{y}_{..}\right) + \sum_{i=1}^{v} \tau_i Q_i$$

is the sum of squares due to block ignoring treatments, *i.e.*, it is (unadjusted) block sum of squares.

The analysis of variance table for Intrablock analysis of BIBD is given in *Table 6·77*.

TABLE 6·77 : ANOVA FOR B.I.B.D. (INTRABLOCK ANALYSIS)

Source	*d.f.*	*Sum of Squares*	*E (M.S.S.)*
Between blocks (unadjusted)	$b-1$	$\frac{1}{k}\sum_{j=1}^{b} B_j^2-(G^2/bk)$	
Between treatments (adjusted)	$v-1$	$\sum_{i=1}^{v}(Q_i^2/rE)$	$\sigma_e^2+rE\sum_{i=1}^{v}\frac{\tau_i^2}{v-1}$
Intra block error	$bk-b-v+1$	By subtraction	σ_e^2
Total	$bk-1$	$\sum_{(i,j)\in D} y_{ij}^2-(G^2/bk)$ = R.S.S. – C.F.	

In the above table $G=\sum_{(i,j)\in D} y_{ij}$ is the grand total of all the observations and $\sum_{(i,j)\in D} y_{ij}^2$ is the raw sum of squares (R.S.S.).

For testing the null hypothesis : $H_0: t_1=t_2=\ldots=t_v$,
the treatments (adjusted) mean square is compared with the intrablock error mean square as usual.

If we are interested to make inferences about the block effects, the (*adjusted*) block mean sum of squares is to be compared with error mean square. The adjusted block sum of squares can be computed from the following identity :

Block S.S. (adjusted) + Treatment S.S. (unadjusted)
= Block S.S. (unadjusted) + Treatment S.S. (adjusted) ...(6·301)

where, Treatment S.S. (unadjusted) $=\frac{1}{r}\sum_{i=1}^{v} T_i^2-$ C.F. ...(6·301*a*)

Example 6·21. *The data in Table 6·78 gives the results of an experiment for comparing 7 treatments in 7 blocks of 3 units each, there thus being 3 replications of each treatment. Analyse the data.*

Treatments	*Blocks*						
	1	*2*	*3*	*4*	*5*	*6*	*7*
1	50	42	91	—	—	—	—
2	—	—	118	94	94	—	—
3	76	—	—	64	—	80	—
4	—	—	72	—	—	53	31
5	44	—	—	—	65	—	54
6	—	102	—	—	119	92	—
7	—	38	—	38	—	—	37

Solution. The above design is a BIBD with parameters : $v = 7,\ b = 7,\ \ r = 3 = k,\ \lambda = 1$

To carry out the analysis of the design we compute the following quantities :

T_i = Total yield for the ith treatment from all the blocks, $(i = 1, 2, \ldots, 7)$

B_j = Total yield from the jth block, $(j = 1, 2, \ldots, 7)$

$\sum\limits_j n_{ij} B_j$ = Total yield from all the blocks in which ith treatment occurs [$(i = 1, 2, \ldots, 7)$

For example,

$\Sigma n_{1j} B_j$ = Total yield from all the blocks in which 1st treatment occurs.

$= B_1 + B_2 + B_3 = 170 + 182 + 281 = 633$

$\Sigma n_{3j} B_j = B_1 + B_4 + B_6 = 170 + 196 + 225 = 591$; and so on

$$Q_i = T_i - \sum_{j=1}^{7} n_{ij} B_j / k ; \quad (i = 1, 2, \ldots, 7)$$

TABLE 6·78A : CALCULATIONS FOR VARIOUS S.S.

	Treatment Totals' T_i	B_j	B_j^2	$\sum_j n_{ij} B_j$	$\sum n_{ij} B_j/k$	Q_i	Q_i^2	T_i^2
(1)	(2)	(3)	(4)	(5)	(6)	(7) = (2) – (6)	(8)	(9)
1	183	170	28,900	633	211·00	–28·00	781·00	33,489
2	306	182	33,124	755	251·67	54·33	2,951·75	93,636
3	220	281	78,961	591	197·00	23·00	529·00	48,400
4	156	196	38,416	628	209·33	–53·33	2,844·09	24,336
5	163	278	77,284	570	190·00	–27·00	729·00	26,569
6	313	225	50,625	685	228·33	84·67	7,169·00	97,969
7	113	122	14,844	500	166·67	–53·67	2,880·47	12,769
Total	1,454	1,454	3,22,194				17,887·31	3,37,168

If y_{ij} denotes the yield from the experimental unit receiving the ith treatment in the jth block,

$$G = \sum_{(i,j) \in D} y_{ij} = 1{,}454 \quad ; \quad C.F. = \frac{G^2}{21} = \frac{(1{,}454)^2}{21} = \frac{21{,}14{,}116}{21} = 1{,}00{,}672{\cdot}19$$

$$\text{Raw S.S.} = \sum_{(i,j) \in D} y_{ij}^2 = 1{,}15{,}730 \quad ; \quad \text{Total S.S.} = \text{Raw S.S.} - \text{C.F.} = 15{,}057{\cdot}81$$

$$\text{Blocks (unadjusted) S.S.} = \frac{1}{3} \sum_{j=1}^{7} B_i^2 - \text{C.F.} = 1{,}07{,}398{\cdot}00 - 1{,}00{,}672{\cdot}19 = 6{,}725{\cdot}81$$

$$\text{Treatments (adjusted) S.S.} = \sum_{i=1}^{7} Q_i^2/rE_i \qquad [(c.f. \ (6{\cdot}299)]$$

where $E = \dfrac{\lambda v}{kr} \quad \Rightarrow \quad rE = \dfrac{\lambda v}{k} = \dfrac{7}{3}$

$$\therefore \ \text{Treatments (adjusted) S.S.} = \frac{17{,}887{\cdot}31 \times 3}{7} = 7{,}665{\cdot}99$$

Error (Intrablock) S.S. = Total S.S. – Blocks (unadjusted) S.S. – Treatments (adjusted) S.S.
= 15,057·81 – 6,725·81 – 7,665·99 = 666·01

TABLE 6·79 : ANOVA TABLE FOR BIBD (INTRA-BLOCK ANALYSIS)

Source of Variation	*S.S.*	*d.f.*	*Mean S.S.*	*Variance Ratio*
Blocks (Unadjusted)	6,725·81	6	1,120·97	$\frac{1{,}120{\cdot}97}{83{\cdot}25} = 13{\cdot}46$
Treatments (Adjusted)	7,665·99	6	1,277·67	$\frac{1{,}277{\cdot}67}{83{\cdot}25} = 15{\cdot}347^*$
Error	666·01	8	83·25	—
Total	15,057·80	20		

Since tabulated $F_{6,8}$ (0·05) = 3·58, the variance ratio due to treatments is highly significant and hence the null hypothesis

$$H_i : t_1 = t_2 = \ldots = t_7$$

i.e., the treatment effects are equal is rejected at 5% level of significance. Hence, we conclud that the treatments differ significantly from each other.

To find out which pair of treatments differ significantly